MATH 245 F16, Exam 2 Solutions

1. Carefully define the following terms: Proof by Contradiction theorem, Uniqueness Proof theorem, proof by strong induction, Set S is *well-ordered* by $<$.

   The Proof by Contradiction theorem says that for propositions p, q, if $p \wedge \neg q$ is false, then $p \rightarrow q$ is true. The Uniqueness Proof theorem says that if for all x, y in domain D, $P(x) \wedge P(y) \rightarrow x = y$, then predicate P holds for at most one x in the domain. To prove the proposition $\forall x \in \mathbb{N}, P(x)$ by strong induction, we need to prove that $P(1)$ is true (base case), and that for any $k \in \mathbb{N}$, that $P(1) \wedge P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$. (inductive case). Set S is well-ordered by $<$ if every subset of S contains an element that is minimal with respect to $<$.

2. Carefully define the following terms: recurrence, $a_n = \Theta(b_n)$, $S = T$ (for sets S, T), $S \cup T$ (for sets S, T).

   A recurrence is a sequence of numbers, all but finitely many of whose terms are defined in terms of its previous terms. $a_n = \Theta(b_n)$ means that $a_n = O(b_n) \wedge a_n = \Omega(b_n)$. $S = T$ if the sets S, T contain exactly the same elements. $S \cup T$ is the set $\{x : x \in S \vee x \in T\}$.

3. Let $n \in \mathbb{Z}$. Prove that $\frac{(n+1)(n-2)}{2} \in \mathbb{Z}$.

   We apply the division algorithm to $n, 2$ to get $q, r \in \mathbb{Z}$ with $n = 2q + r$ and $0 \le r < 2$. The proof continues in two cases. If $r = 0$ then $\frac{(n+1)(n-2)}{2} = (n+1)\frac{2q+0-2}{2} = (n+1)(q-1) \in \mathbb{Z}$. If instead $r = 1$ then $\frac{(n+1)(n-2)}{2} = (n-2)\frac{2q+1+1}{2} = (n-2)(q+1) \in \mathbb{Z}$.

4. Use mathematical induction to prove that $\forall n \in \mathbb{Z}$ with $n \ge 3$, $2^n > 5$.

   Base case: $n = 3$. $2^n = 8 > 5$, done.
   Inductive case: Let $n \in \mathbb{Z}$ with $n \ge 3$, and assume that $2^n > 5$. Multiply both sides by 2 to get $2^{n+1} = 2 \cdot 2^n > 2 \cdot 5 = 10 > 5$. Hence $2^{n+1} > 5$.

5. Suppose that an algorithm has runtime specified by the recurrence relation $T_n = n^{1/2} T_{n/2} + 2$. Determine what, if anything, the Master Theorem tells us.

   Applying the Master Theorem, we find $a = n^{1/2}$, $b = 2$, $c_n = 2$. Since a is not a constant, then the Master Theorem does not apply. It tells us nothing.

6. Let S, T be sets with $S \cap T = S$. Prove that $S \subseteq T$.

   Let $x \in S$. Since $S \cap T = S$, S and $S \cap T$ have the same elements; in particular, $x \in S \cap T$. Hence $x \in S \wedge x \in T$. By simplification, $x \in T$. This proves that $S \subseteq T$.

7. Let S be a set. Prove that $S \setminus \emptyset = S$.

   Let $x \in S \setminus \emptyset$. Then $x \in S \wedge x \notin \emptyset$. By simplification, $x \in S$. This proves that $S \setminus \emptyset \subseteq S$.
   Now, let $x \in S$. Also, $x \notin \emptyset$, since $\emptyset$ contains no elements. Hence, by conjunction, $x \in S \wedge x \notin \emptyset$. Thus $x \in S \setminus \emptyset$. This proves that $S \subseteq S \setminus T$.

8. Let $x \in \mathbb{R}$. Prove that $2\lfloor x \rfloor \le \lfloor 2x \rfloor \le 2\lfloor x \rfloor + 1$.

   Since $x \ge \lfloor x \rfloor$, we have $2x \ge x + \lfloor x \rfloor$. By Theorems 5.16 and 5.17, we have $\lfloor 2x \rfloor \ge \lfloor x + \lfloor x \rfloor \rfloor = \lfloor x \rfloor + \lfloor x \rfloor = 2\lfloor x \rfloor$. This proves the first inequality.
   Since $x < \lfloor x \rfloor + 1$, we have $2x < x + \lfloor x \rfloor + 1$. By Theorems 5.16 and 5.17, we have $\lfloor 2x \rfloor \le \lfloor x + \lfloor x \rfloor + 1 \rfloor = \lfloor x \rfloor + \lfloor x \rfloor + 1 = 2\lfloor x \rfloor + 1$. This proves the second inequality.

9. Let $x \in \mathbb{R}$ with $x > -1$. Prove that $\forall n \in \mathbb{N}_0, (1+x)^n \ge 1 + nx$.

   We use (shifted) induction on n. Base case: $n = 0$. $(1+x)^0 = 1 \ge 1 + 0x$, as desired.
   Inductive case: Let $n \in \mathbb{N}_0$ with $(1+x)^n \ge 1 + nx$. We multiply both sides by $(1+x)$; since this is positive the inequality is preserved. The result is $(1+x)^{n+1} = (1+x)(1+x)^n \ge (1+x)(1+nx) = 1 + nx + x + nx^2 \ge 1 + nx + x = 1 + (n+1)x$.

10. Prove that $3^n \ne O(2^n)$.

    We use proof by contradiction. Suppose that $3^n = O(2^n)$. Then there are $n_0 \in \mathbb{N}$ and $M \in \mathbb{R}$ such that for all $n \ge n_0$, $|3^n| \le M|2^n|$. Set $m = \log_{3/2} M$, and take some $n > \max\{n_0, m\}$. Since $n > n_0$, we have $3^n \le M2^n$, which rearranges to $(3/2)^n \le M$. But also, since $(3/2)^x$ is an increasing function of x, we have $(3/2)^n > (3/2)^m = (3/2)^{\log_{3/2} M} = M$. This is a contradiction.